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A Necessary and Sufficient Condition for A Simplicial Complex to be An Order Complex

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A partially ordered set is also called a poset. For some years past we have been concentrating our attention on the number of chains in a poset and the roles which the chains play in the study of combinatorial quantities and qualities in mathematics. The purpose of this talk is to clarify the scope of Euclidean simplicial complexes grasped by the "poset-chain" structure. In this talk, posets and complexes are restricted to finite poset and finite complexes. It is well known that there is a one to one correspondence between the Euclidean simplicial complexes and the abstract simplicial complexes, and that as a typical example of an abstract simplicial complex there is an order complex. An order complex $\Delta(P)$ is one induced by a poset P, in which each chain itself in P is a simplex in $\Delta(P)$. This means that the number of simplexes in $\Delta(P)$ is completely determined by the chain polynomial of P, whose coefficient of x^k is the number of chains of length k in P. The chain polynomial (also called the command flow polynomial) of a poset is fully investigated in [1-3, 5, 7]. Therefore, if we get the precise relationships between the Euclidean simplicial complexes and the order complexes, then we easily know the combinatorial quantities and qualities related to the number of simplexes of a Euclidean complex. Generally, each simplicial complex is not necessarily an order complex. By a diagram, we first summarize the easily known relationships between the non-isomorphic abstract simplicial complexes $\mathcal{A}_c$, the non-isomorphic Euclidean simplicial complexes $\mathcal{E}_c$ and the non-isomorphic posets $\mathcal{P}_s$. In Fig.1, the pair(ϕ,ψ) of mappings is a one to one correspondence defined by $\psi(K) = \Delta(K)$ (the abstract complex of a Euclidean complex K) and $\phi(\Delta) = g(\Delta)$ (the geometric realization of an abstract complex Δ), the mapping β defined by $\beta(P) = \Delta(P)$ is neither onto nor one to one, and so is the mapping γ defined by $\gamma(P) = g(\Delta(P))$. Note that $\phi\psi$ = an identity, $\beta = \psi\gamma$ and $\gamma = \phi\beta$.

$$\begin{tikzcd}
\mathcal{A}_c \arrow[rr, shift left, "\phi"] & & \mathcal{E}_c \arrow[ll, shift left, "\psi"] \\
& \mathcal{P}_s \arrow[ul, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Fig. 1

We next state a characterization for the family $\gamma(\mathcal{P}s) = \{g(\Delta(P)) \mid P \in \mathcal{P}s\}$. Let K be any Euclidean simplicial complex. The <u>skelton</u> $K^{(1)}$ of K is a simple graph (V(K), E(K)) composed of the 0-simplexes V(K) and the 1-simplexes E(K) of K, in which V(K) and E(K) are called the <u>vertex-set</u> and the <u>edge-set</u> of K, respectively.

Definition. Let K be a Euclidean simplicial complex. Then an orientaion α on the edge-set E(K) of the skelton $K^{(1)}$ is said to be <u>simplicial</u> if α satisfies the following conditions:

(1) the relation $\tilde{\alpha}$ on the vertex-set V(K) induced by α, that is, $\tilde{\alpha}$: u > v iff α: u → v, with the added reflexive relation, comes to a partial ordering on V(K),

(2) for any chain $v_0 < v_1 < \cdots < v_k$ ($k \geqq 0$) in the poset ($V(K)$, $\tilde{\alpha}$), there exists a k-simplex $\langle v_0, v_1, \cdots, v_k \rangle$ ($k \geqq 0$) in K.

Theorem. <u>For any Euclidean simplicial complex K, there exists a poset P satisfying</u> $\Delta(K) \cong \Delta(P)$ <u>if and only if there exists a simplicial orientation on the edge-set</u> E(K) <u>of the skelton</u> $K^{(1)}$.

It is noted that in showing the sufficiency, we use an elementary theorem in graph theory that there exists a Hamilton path in a complete digraph. The details and further informations of this talk are described in [8].

Remark. The faces of every polyhedral complex K form a poset under inclusion, which is called the <u>face-poset</u> of K and denoted by P(K). It is well known that the geometric realization $g(\Delta(P(K)))$ of the order complex $\Delta(P(K))$ is isomorphic to the <u>first barycentric subdivision</u> K' of K. This fact means that there exists a simplicial orientation on the edge-set E(K') of the skelton $K'^{(1)}$, that is, the family {K' | K is a polyhedral complex} is contained in the family $\gamma(\mathcal{P}s)$.

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